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Best Multipoint Local L_p Approximation

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Introduction

Let M be a finite-dimensional subspace of $C(I)$ where $I = [a, b]$ and let $X = \{x_1, ..., x_k\}$ where $a \leqslant x_1 < \cdots < x_k \leqslant b$ and $k \geqslant 1$. Let $f \in C(I)$ be fixed. Then for each fixed $1 \leqslant p \leqslant \infty$ and for all positive and sufficiently small h, there exists at least one $q_h \in M$ that minimizes

$$\sum_{j=1}^{k} \int_{x_j}^{x_j+h} |f(t) - q(t)|^p \, dt \qquad \text{as} \quad q \text{ ranges over } M. \tag{1.1}$$

(If $p = \infty$ we consider $\max_{1 \leqslant j \leqslant k} \{\max_t |f(t) - q(t)|: x_j \leqslant t \leqslant x_j + h\}$).

We call $q^* \in M$ a best local k-point approximation to f if there is a sequence $h_\nu \to 0^+$ such that $q_{h_\nu} \to q^*$. The purpose of this paper is to study the existence, uniqueness, and characterization question for this problem in the case where M is an $n+1$ dimensional extended Tchebycheff subspace of $C[a, b]$. We are able to show that for $1 < p \leqslant \infty$ and $f \in C^n[a, b]$ a best k-point local approximant exists, is unique, and is characterized as the solution of a certain optimization problem involving only the values of f and its derivatives up to a certain order (depending on n and k) at the points $x_1, ..., x_k$. The results obtained may be regarded as providing a natural way of extending the classical interpolation theory of polynomials (including Taylor's polynomials and Hermite interpolation) to situations

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where they do not normally apply (i.e., when k does not necessarily divide $n+1$).

The case $k=1$ was studied in the papers [1 3] while the case $k=2$ was considered in [4]. Beatson and Chui introduced the general multipoint local approximation problem in [5] and obtained partial existence and characterization results in special cases. We shall refer to these results later.

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Definitions and Notation

Throughout this paper, I will denote the interval $[a, b]$, n and k will be fixed positive integers with $k \leqslant n+1$, and X will denote the fixed set $\{x_1, ..., x_k\}$ where $a \leqslant x_1 < \cdots < x_k \leqslant b$. The integers l and r will be defined by

$$l = \left[\frac{n+1}{k}\right] \qquad \text{and} \qquad r = n+1-lk, \tag{2.1}$$

where [ ] denotes the greatest integer function. The set $\{U_i\}_{i=0}^{n} \subset C^n(I)$ will be an ETS of order l and M will denote span $\{U_0, ..., U_n\}$. (Recall that the order of $\{U_i\}_{i=0}^{n} = l$ means that if $z_1, ..., z_m$ are distinct points in $[a, b]$ and if $j_1, ..., j_m$ are nonnegative integers such that $j_i \leqslant l-1$, $i=1, ..., m$, and $j_1 + \cdots + j_m = n+1$, then there is a unique $q \in M$ such that $q^{(i)}(z_s) = f^{(i)}(z_s)$, $i=0, ..., j_s$, $s=1, ..., m$.)

For h satisfying $0 < h \leqslant \min_{1 \leqslant j \leqslant k-1} |x_{j+1} - x_j|$ let I_h denote $\bigcup_{j=1}^{k} [x_j, x_j + h]$.

Given $f \in C^{l-1}[I]$ we define

$$S = \{q \in M: q^{(i)}(x_j) = f^{(i)}(x_j),\ i=0, ..., l-1;\ j=1, ..., k\} \tag{2.2}$$

$$N(g) = \left[\sum_{i=0}^{l} \sum_{j=1}^{k} |g^{(i)}(x_j)|^P\right]^{1/P}, \qquad g \in C^l[X] \tag{2.3}$$

$$N_h(g) = \left[\sum_{j=1}^{k} \frac{C_p}{h^{lP+1}} \int_{x_j}^{x_j+h} |g(t)|^P \, dt\right]^{1/P} \qquad (1 \leqslant P < \infty),\ g \in L_P(I_h), \tag{2.4}$$

where C_p is a constant independent of h, g, and f to be specified later. In the case $P = \infty$ we define

$$N_h(g) = C_\infty \max_{1 \leqslant j \leqslant k} \max_{t \in [x_j, x_j+h]} |g(t)|, \qquad g \in C(I_h).$$

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As stated in the Introduction we wish to consider the behavior of $\{q_h\}$ as $h \to 0^+$ where q_h minimizes (1.1). Our first task is to show that for appropriate f the net $\{q_h\}$ in fact has at least one cluster point as $h \to 0^+$. Since $q_h = f$ for at least $n+1$ points in $[a, b]$ (see Lemma 1) we shall analyze this problem by considering the properties of interpolating "polynomials" following the approach in [3].

Let $X^* = \{y_1, ..., y_r\} \subset I$ be such that $y_1 < \cdots < y_r$ and $r \leq n+1$. For each v, let $\{x_{ij}(v)\}$, $j = 1, ..., m_i$, be sequences in I satisfying

$$a \leq x_{11}(v) < \cdots < x_{1m_1}(v) < x_{21}(v) < \cdots < x_{2m_2}(v) < \cdots < x_{r1}(v) < \cdots < x_{rm_r}(v) \leq b \tag{3.1}$$

$$\lim_{v \to \infty} x_{ij}(v) = y_i, \qquad i = 1, ..., r,\ j = 1, ..., m_i, \tag{3.2}$$

where each m_i is an integer greater than or equal to 1 with $\sum_{i=1}^r m_i = n+1$.

Now, given $f \in C(I)$, let q_v denote the unique element of M that satisfies

$$q_v(x_{ij}(v)) = f(x_{ij}(v)), \qquad i = 1, 2, ..., r,\ j = 1, 2, ..., m_i. \tag{3.3}$$

LEMMA 1. *For each $i = 0, 1, ..., n$ the coefficients $a_i(v)$ of q_v can be written in the form*

$$a_i(v) = \frac{\begin{vmatrix} u_0[x_{11}] \cdots\cdots & f[x_{11}] & \cdots\cdots u_n[x_{11}] \\ \vdots & \vdots & \vdots \\ u_0[x_{11}, ..., x_{1m_1}] \cdots & f[x_{11}, ..., x_{1m_1}] & \cdots u_n[x_{11}, ..., x_{1m_1}] \\ u_0[x_{21}] \cdots\cdots & f[x_{21}] & \cdots\cdots u_n[x_{21}] \\ \vdots & \vdots & \vdots \\ u_0[x_{21}, ..., x_{2m_2}] \cdots & f[x_{21}, ..., x_{2m_2}] & \cdots u_n[x_{21}, ..., x_{2m_2}] \\ \vdots & \vdots & \vdots \\ u_0[x_{r1}] \cdots\cdots & f[x_{r1}] & \cdots\cdots u_n[x_{r1}] \\ \vdots & \vdots & \vdots \\ u_0[x_{r1}, ..., x_{rm_r}] \cdots & f[x_{r1}, ..., x_{rm_r}] & \cdots u_n[x_{r1}, ..., x_{rm_r}] \end{vmatrix}}{\begin{vmatrix} u_0[x_{11}] \cdots\cdots & u_i[x_{11}] & \cdots\cdots u_n[x_{11}] \\ \vdots & \vdots & \vdots \\ u_0[x_{11}, ..., x_{1m_1}] \cdots & u_i[x_{11}, ..., x_{1m_1}] & \cdots u_n[x_{11}, ..., x_{1m_1}] \\ \vdots & \vdots & \vdots \\ u_0[x_{r1}] \cdots\cdots & u_i[x_{r1}] & \cdots\cdots u_n[x_{r1}] \\ \vdots & \vdots & \vdots \\ u_0[x_{r1}, ..., x_{rm_r}] \cdots & u_i[x_{r1}, ..., x_{rm_r}] & \cdots u_n[x_{r1}, ..., x_{rm_r}] \end{vmatrix}}.$$

Proof. Applying Cramer's rule to (3.3) and using row operations and elementary properties of determinants the lemma follows immediately. ∎

Now by elementary properties of divided differences

$$\lim_{\nu \to \infty} g[x_{i1}, x_{i2}, ..., x_{im_i}] = \frac{g^{(m_i - 1)}(y_i)}{(m_i - 1)!}, \qquad i = 1, ..., r, \tag{3.4}$$

if $g^{(m_i - 1)}(x)$ is continuous at y_i, $i = 1, ..., r$.

We now have

THEOREM 1. *Let $f \in C^m[I]$ where $m = \max_{1 \leqslant i \leqslant r} m_i$ (where M is an extended Tchebycheff space of dimension $n+1 \geqslant m$) and let $\{x_{ij}(\nu)\}$ and $\{q_\nu\} \subset M$ satisfy* (3.1), (3.2), *and* (3.3). *Then the sequence $\{q_\nu\}$ is uniformly bounded on I and converges uniformly to $q_0 \in M$ which satisfies*

$$q_0^{(j)}(y_i) = f^{(j)}(y_i), \qquad i = 1, ..., r \text{ and } j = 0, 1, ..., m_i - 1. \tag{3.5}$$

Proof. Lemma 1 implies that the coefficients $a_i(\nu)$ converge as $\nu \to \infty$ to coefficients a_i, $i = 1, ..., r$, which by (3.4) are exactly the solutions obtained when Cramer's rule is applied to (3.5). ∎

COROLLARY 1. *Let $f \in C^n[I]$ and for each $h > 0$ let q_h denote a (unique for $1 < p \leqslant \infty$) L^p approximation to f on I_h by elements of M. Let $\{q_{h_\nu}\}$ be an arbitrary subsequence of $\{q_h\}$. Then $\{q_{h_\nu}\}$ is uniformly bounded and hence has at least one cluster point q_0 as $\nu \to \infty$. Moreover, q_0 satisfies* (3.5) *for some appropriate set of y_i's.*

In addition to Theorem 1, the proof of the corollary rests on the following well known property of ETSs.

LEMMA 2. *Let $f \in C[I]$ and q^* be any element of M such that $f - q^*$ has at most n sign changes in I. Then there exists a $q \in M$ such that $q \neq 0$ and $(f - q^*)q \geqslant 0$ on I.*

Proof of Corollary 1. For simplicity we will write q_ν and I_ν instead of q_{h_ν} and I_{h_ν}, respectively. In view of Theorem 1 it suffices to show that $f - q_\nu$ has at least $n+1$ zeros on I. Suppose not. Then there exists $q \neq 0$ in M such that $(f - q_\nu)q \geqslant 0$ on I. Assume $1 < P < \infty$. (The proof for the other cases is similar.) Then

$$\int_{I_\nu} |f - q_\nu|^{P-1} \operatorname{sgn}(f - q_\nu) m \, d\mu_\nu = 0 \qquad \text{for all} \quad m \in M, \tag{3.6}$$

where $\mu_\nu = \mu\chi(I_\nu)$, μ is Lebesgue measure, and $\chi(I_\nu)$ is the indicator function of I_ν. Applying (3.6) for $m = q$ we conclude that

$$\int_{I_\nu} |f - q_\nu|^{p-1} \, |q| \, d\mu_\nu = 0$$

which is a contradiction since the integrand may vanish on a set of measure zero only and is nonnegative. ∎

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K-Point Local Approximation $(1 \leqslant P < \infty)$

The notation and setting are as in (2.1)–(2.4).

LEMMA 3. *Let* $f \in C^{l+1}[I]$ *and let* $1 \leqslant P < \infty$. *Then*

$$\int_{I_h} |f - q|^P \, dt \leqslant O(h^{Pl+1}) \qquad \text{for every} \quad q \in S. \tag{4.1}$$

Proof. Let $q \in S$ and let $\varphi = f - q$. Then using the Taylor expansion of φ about x_j and using the definition of S we have

$$\varphi(t) = \frac{(t - x_j)^l}{l!} \varphi^{(l)}(x_j) + O((t - x_j)^{l+1})$$

and hence

$$\int_{I_h} |\varphi(t)|^P \, dt = \sum_{j=1}^{k} \int_{x_j}^{x_j + h} \left| \frac{(t - x_j)^l}{l!} \varphi^{(l)}(x_j) + O((t - x_j)^{l+1}) \right|^P dt. \tag{4.2}$$

Changing variables on the right hand side of (4.2) we get

$$\begin{aligned} \int_{I_h} |\varphi(t)|^P &= \sum_{j=1}^{k} h \int_0^1 \left| \frac{h^l u^l}{l!} \varphi^{(l)}(x_j) + O((hu)^{l+1}) \right|^P du \\ &= \sum_{j=1}^{k} h^{Pl+1} \int_0^1 \left| \frac{\varphi^{(l)}(x_j)}{l!} u^l + O((hu)^{l+1}) \right|^P du \\ &\leqslant O(h^{Pl+1}). \end{aligned}$$ ∎

THEOREM 2. *For each* $h > 0$, *let* q_h *be a best* L^P $(1 \leqslant p < \infty)$ *approximation to* $f \in C^{l+1}[I_h]$ *from* M. *Suppose that* $q_h \to q_0$ *as* $h \to 0^+$. *Then* $q_0 \in S$ *so that*

$$q_0^{(i)}(x_j) = f^{(i)}(x_j), \qquad i = 0, ..., l-1, \; j = 1, ..., k.$$

Proof. From Lemma 3 and the definition of q_h we have that

$$\sum_{j=1}^{k} \int_{x_j}^{x_j+h} |(f-q_h)(t)|^P \, dt \leqslant O(h^{Pl+1})$$

and hence

$$\int_{x_j}^{x_j+h} |(f-q_h)(t)|^P \, dt \leqslant O(h^{Pl+1}), \qquad j=1, ..., k.$$

Now without loss of generality consider the case $\int_0^h |(f-q_h)(t)|^P \, dt \leqslant O(h^{Pl+1})$ and suppose that as $h \to 0^+$, $q_h \to q_0$ such that

$q_0^{(i)}(0) = f^{(i)}(0)$, $i=0, 1, ..., s-1$, where $s < l$ and $q_0^{(s)}(0) \neq f^{(s)}(0)$, where negative superscripts are suppressed if $s=0$. (4.3)

Letting $E_h = f - q_h$ for $h \geqslant 0$ we have after the change of variable $hu = t$,

$$\int_0^1 |E_h(hu)|^P \, du \leqslant O(h^{Pl}).$$

Using the Taylor expansion of $E_h(hu)$ about $u=0$ with exact remainder (see Davis [6]) we get

$$\int_0^1 \left| E_h(0) + huE_h^1(0) + \cdots + \frac{h^{s-1}u^{s-1}}{(s-1)!} E_h^{(s-1)}(0) + E_h[0, 0, ..., 0, hu]\, h^s u^s \right|^P du \leqslant O(h^{Pl}).$$

This may be written in the equivalent form

$$\int_0^1 |A_0(h) + A_1(h)u + \cdots + A_{s-1}(h)u^{s-1} + A_s(h, u)u^s|^P \, du \leqslant O(h^{P(l-s)}), \tag{4.4}$$

where $A_i(h) = E_h^{(i)}(0)/h^{s-i} i!$, $i = 0, 1, ..., s-1$, and $A_s(h, u) = E_h[0, 0, ..., 0, hu]$.

Taking limits in (4.3) as $h \to 0^+$ we must consider two cases:

Case 1. Each $A_i(h)$, $i=0, 1, ..., s-1$, is bounded as $h \to 0^+$. Then by going to an appropriate subsequence $h_v \to 0^+$ we obtain

$$0 = \int_0^1 \left| A_0 + A_1 u + \cdots + A_{s-1} u^{s-1} + \frac{E_0^{(s)}(0)}{s!} u^s \right|^P du$$

which is a contradiction since $E_0^{(s)}(0) \neq 0$.

Case 2. There is a sequence $\{h_\nu\}$ such that $N_\nu \equiv \max_{0 \leqslant i \leqslant s-1} |A_i(h_\nu)|^P \to \infty$ as $h_\nu \to 0^+$. Dividing both sides of (4.4) by N_ν and going to a subsequence if necessary we obtain the equality $0 = \int_0^1 |B_0 + B_1 u + \cdots + B_s u^s|^P \, du$ where $\max_{0 \leqslant i \leqslant s} |B_i| = 1$ which is also a contradiction. Thus (4.3) leads to a contradiction and hence $f^{(i)}(0) = q_0^{(i)}(0)$, $i = 0, 1, ..., s-1$, where $s \geqslant l$. Thus $q_0^{(i)}(x_j) = f^{(i)}(x_j)$, $i = 0, ..., l-1$, $j = 1, ..., k$. ▌

COROLLARY 2. *Let $n = lk - 1$ and let $f \in C^{l+1}[I]$ and $1 < P < \infty$ be fixed. For each $h > 0$ let q_h denote the (unique) best L^P approximation to f on I_h by elements of M. Then the net $\{q_h\}$ converges uniformly to the unique element $q_0 \in M$ satisfying*

$$q_0^{(i)}(x_{ij}) = f^{(i)}(x_j), \qquad i = 0, ..., l-1, j - 1, ..., k. \tag{4.5}$$

Proof. If $n + 1 = lk$, then the set S contains exactly one element, namely the Hermite interpolating "polynomial" satisfying (4.5), and hence by Theorem 2 and Corollary 1, we have the result. ▌

Remark. Corollary 2 was proved in a different manner by Beatson and Chui in [5].

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Characterization of Local L^P Approximants

We shall now characterize the properties of local approximants. We shall need the following definition.

DEFINITION. For any interval $[\alpha, \beta]$ and any fixed P $(1 \leqslant P \leqslant \infty)$ the l distinct points $\{t_1^*, ..., t_l^*\} \subset [\alpha, \beta]$ are called the Tchebycheff points for $[\alpha, \beta]$ if they minimize the quantity

$$E(t_1, ..., t_l) = \left\| \prod_{i=1}^{l} (\cdot - t_i) \right\|_P \qquad \text{over all} \quad (t_1, ..., t_l) \subset [\alpha, \beta]^l.$$

Remark. From well known results in approximation theory (see Davis [6], for example) the Tchebycheff points in (0, 1) exist and are unique for each P, $1 \leqslant P \leqslant \infty$. If we denote these by $\{u_1^*, ..., u_l^*\}$ then the corresponding Tchebycheff points in (α, β) are given uniquely by the relationship

$$t_i^* = (\beta - \alpha) u_i^* + \alpha, \qquad i = 1, ..., l. \tag{5.1}$$

We now define $\alpha_P = [\int_0^1 |\prod_{i=1}^l (u - u_i^*)|^P \, du]^{1/P}$, $1 \leqslant P < \infty$, and $\alpha_\infty = \max_{u \in [0,1]} |\prod_{i=1}^l (u - u_i^*)|$.

In view of (5.1), for each $h>0$ and each j, $1 \leqslant j \leqslant k$, there exist l unique points in (x_j, x_j+h) say $t^*_{1j}(h), ..., t^*_{lj}(h)$ such that $u^*_i = (t^*_{ij}(h) - x_j)/h$, $i = 1, ..., l$. (These being the Tchebycheff points for $[x_j, x_j+h]$.) Now let $q \in S$ be arbitrary and for each $h>0$ define $\hat{q}_h \in M$ by the following interpolation condition:

(i) $\hat{q}_h(t^*_{ij}(h)) = f(t^*_{ij}(h))$, $i = 1, ..., l$, $j = 1, ..., k$.

(ii) $\hat{q}_h(t) = q(t)$ at any r points distinct from the x_j's. (Recall $n+1 = r+lk$.)

Then from Theorem 1 we infer that $\hat{q}_h \to q$ as $h \to 0$. Moreover we have the following relationship between $\hat{q}_h$ and q.

LEMMA 4. *Assume $f \in C^{l+1}[I]$ and $1 \leqslant P < \infty$. Let $q \in S$ be arbitrary and let $\hat{q}_h$ be defined by* (i) *and* (ii). *Then $N_h(f - \hat{q}_h) \to N(f-q)$ as $h \to 0$ where $C_P = (l!/\alpha_P)^P$ in the definition of $N_h(\cdot)$ (see* (2. 4)).

Proof.

$$N^P_h(f - \hat{q}_h) = \sum_{j=1}^{k} \frac{C_P}{h^{lP+1}} \int_{x_j}^{x_j+h} |\hat{E}_h[t^*_{ij}(h), ..., t^*_{lj}(h), t]|^P \times \prod_{i=1}^{l} \left| t - t^*_{ij}(h) \right|^P dt,$$

where $\hat{E}_h(t) = (f - \hat{q}_h)(t)$ and where we have used the fact that the Newton interpolating polynomial of degree $\leqslant l-1$ that interpolates $\hat{E}_h(t)$ at $t^*_{ij}(h), ..., t^*_{lj}(h)$ is identically zero. Using the mean value theorem for integrals we get

$$N^P_h(f - q_h) = \sum_{j=1}^{k} \left\{ \frac{C_P}{h^{lP+1}} |\hat{E}_h[t^*_{ij}(h), ..., t^*_{lj}(h), x_j + \beta_{ij}(h)]|^P \times \int_{x_j}^{x_j+h} \left| \prod_{i=1}^{l} (t - t^*_{ij}(h)) \right|^P dt \right\},$$

where $0 < \beta_{ij} < 1$, $i = 1, ..., l$, $j = 1, ..., k$.

By changing variables in each inverval separately using $u = (t - x_j)/h$, $j = 1, ..., k$ we get

$$N^P_h(f - \hat{q}_h) = \sum_{j=1}^{k} C_P |\hat{E}_h[t^*_{ij}(h), ..., t^*_{lj}(h), x_j + \beta_{ij} h]|^P \times \int_0^1 \left| \prod_{i=1}^{k} (u - u^*_i) \right|^P du$$

so as $h \to 0^+$, $N_h^P(f-\hat{q}_h) \to \sum_{j=1}^k |E_0^{(l)}(x_j)|^P$ *where* $E_0(t)=f(t)-q(t)$. Thus $\lim_{h\to 0^+} N_h(f-\hat{q}_h)=N(f-q)$. ▌

Remark. Of course it would be preferable to be able to prove Lemma 4 using q_h instead of $\hat{q}_h$. The difficulty that arises is the possibility that $f-q_h$ might not have a full set of roots in each $[x_j, x_j+h]$ and that it does not seem easy to prove directly that the appropriate number of roots cluster at each x_j as $h\to 0$. Hence we use the auxiliary net $\hat{q}_h$.

COROLLARY 3. *Let $f \in C^{l+1}(I)$. If q_h is a best L_P approximation to f from M on I_h and if $q_h \to q_0 \in S$ as $h \to 0^+$ then*

$$\limsup_{h\to 0^+} N_h(f-q_h) \leqslant N(f-q_0).$$

Proof. $N_h(f-q_h) \leqslant N_h(f-\hat{q}_h) \to N(f-q_0)$ as $h\to 0^+$, which clearly yields the corollary. ▌

LEMMA 5. *Let $f\in C^{l+1}[I]$ and for each $h>0$ let q_h be a best L_P approximation to f on I_h from M. If $q_h \to q_0 \in S$ as $h\to 0^+$ then*

$$\liminf_{h\to 0^+} N_h(f-q_h) \geqslant N(f-q_0).$$

Proof.

$$\begin{aligned} N_h^P(f-q_h) &= \sum_{j=1}^k \frac{C_P}{h^{Pl+1}} \int_{x_j}^{x_j+h} |(f-q_h)(t)|^P\, dt \\ &= \sum_{j=1}^k \frac{C_P}{h^{lP}} \int_0^1 |E_h(x_j+hu)|^P\, du, \end{aligned}$$

where $E_h(t)=(f-q_h)(t)$ and where the change of variable $u=(t-x_j)/h$ has been made in each integral $j=1, ..., k$. Expanding $E_h(x_j+hu)$ in a Taylor's expansion about $u=0$ and dividing by h^{lP} we get

$$\int_0^1 \frac{|E_h(x_j+hu)|^P}{h^{lP}}\, du = \int_0^1 |A_{0j}(h)+A_{1j}(h)u + \cdots + A_{lj}(h)u^l + O(h)|^P\, du, \tag{5.2}$$

where $A_{ij}(h)=E_h^{(i)}(x_j)/i!\, h^{l-i}$, $i=0, ..., l$, $j=1, ..., k$.

Claim. $\max_{0\leqslant i\leqslant 1} |A_{ij}(h)| \equiv M_j(h)$ is bounded as $h\to 0^+$.

Proof. If not there is a sequence $h_\nu \to 0^+$ such that $M_j(h_\nu)\to\infty$. Since (5.2) is bounded ($N_h^P(f-q_h) \leqslant N_h^P(f-q)$ for $q\in S$ and $N_h^P(f-q)$ is bounded as $h\to 0$ by Lemma 3) then dividing both sides by $M_j(h_\nu)$

and going to a subsequence if necessary we arrive at the contradiction $0 = \|q\|_P$, where $q \in \Pi_{l-1}$ has at least one coefficient equal to one.

Thus $\{M_i(h)\}$ is bounded and so let $\{h_\nu\} \to 0^+$ be an arbitrary sequence such that $A_{ij}^\nu \equiv A_{ij}(h_\nu) \to A_{ij}$ as $\nu \to \infty$, $i = 0, ..., l$, $i = 1, ..., k$, where

$$A_{lj} = \frac{E_0^{(l)}(x_j)}{l!}, \qquad j = 1, 2, ..., k.$$

Let $W = \{j\colon E_0^{(l)}(x_j) \neq 0\}$ and for $j \in W$ let $B_{ij}^\nu = A_{ij}^\nu / A_{lj}^\nu$. Then

$$\begin{aligned} N_{h_\nu}^P(f - q_h) &= \sum C_P \int_0^1 |A_{0j}^\nu + A_{1j}^\nu u + \cdots + A_{lj}^\nu u^l + O(h_\nu)|^P \, du \\ &\geqslant \sum_{j \in W} C_P \int_0^1 |A_{0j}^\nu + \cdots + A_{lj}^\nu u^l + O(h_\nu)|^P \, du \\ &= \sum C_P |A_{lj}^\nu|^P \int_0^1 |B_{0j}^\nu + \cdots + B_{l-1,j}^\nu u^{l-1} + u^l + O(h_\nu)|^P \, du. \end{aligned}$$

As $\nu \to \infty$ this converges to

$$\sum_{j \in W} C_P \left| \frac{E_0^{(l)}(x_j)}{l!} \right|^P \int_0^1 |B_{0j} + \cdots + B_{l-1,j} u^{l-1} + u^l|^P \, du.$$

Moreover, $\int_0^1 |B_{0j} + \cdots + B_{l-1,j} u^{l-1} + u^l|^P \, du \geqslant \alpha^P$, $j = 1, ..., k$. Thus, $\lim_{\nu \to \infty} N_{h_\nu}(f - q_{h_\nu}) \geqslant \sum_{j \in W} |E_0^{(l)}(x_j)|^P = \sum_{j=1}^k |E_0^{(l)}(x_j)|^P = N^P(f - q_0)$. Starting with a sequence $\{h_\nu\}$ such that $\underline{\lim}_\nu N_{h_\nu}(f - q_{h_\nu}) = \underline{\lim}_{h \downarrow 0^+} N_h(f - q_h)$, and going to subsequences if necessary we conclude $\underline{\lim}_{h \to 0^+} N_h(f - q_h) \geqslant N(f - q_0)$. ∎

We now have the following characterization theorem:

THEOREM 3. *Let $f \in C^{l+1}[I]$ and q_h be a best L^P approximation to f on I_h from M $(1 \leqslant P < \infty)$. If as $h \to 0^+$, $q_h \to q_0 \in S$ then*

(i) $\lim_{h \to 0^+} N_h(f - q_h) = N(f - q_0)$

(ii) $N(f - q_0) \leqslant N(f - q)$ *for all* $q \in S$.

Proof. (i) This follows immediately from Corollary 3 and Lemma 5. Let $q \in S$ be arbitrary and define $\hat{q}_h$ for q as in Lemma 4.

(ii) From Lemma 4, we have $\lim_{h \to 0^+} N_h(f - \hat{q}_h) = N(f - q)$. But since $N_h(f - q_h) \leqslant N_h(f - \hat{q}_h)$ then $N(f - q_0) = \lim_{h \to 0^+} N_h(f - q_h) = \lim_{h \to 0^+} N_h(f - \hat{q}_h) = N(f - q)$. Since q was arbitrary in S we have proved (ii). ∎

Remark. For future reference we note that in the previous results where we have assumed $q_h \to q_0$ as $h \to 0^+$, the results are still valid with the weaker assumption $q_{h_\nu} \to q_0$ where h_ν is some sequence such that $h_\nu \to 0^+$.

6

Existence and Uniqueness of Local Approximants

In view of Theorem 3 it is of interest to determine when the problem

$$\text{Minimize } N(f-q) \text{ as } q \text{ ranges over } S \tag{6.1}$$

has a unique solution. The existence of a solution is clear since $N(f-\cdot)$ is a continuous seminorm and S is a translate of the finite dimensional subspace of M, $S_0 := \{q \in M/q^{(i)}(x_j) = 0, \quad i = 0, ..., \quad l = 1; \quad j = 1, ..., k\}$. For $1 < p < \infty$ we have the following.

THEOREM 4. *Given $f \in C^{l+1}[I]$, there is a unique $q_0 \in S$ solving* (6.1) *for each $1 < p < \infty$.*

Proof. For $q \in S$ define Φ: $C^{l+1}[I] \to R_k$ by

$$\Phi[g] = [\varphi_1(g), ..., \varphi_k(g)]^T,$$

where $\varphi_j(g) = g^{(l)}(x_j)$, $j = 1, ..., k$, and let $K = \{\Phi(f-q): q \in S\}$. Then K is a closed convex subset of R_k since Φ is a linear map and $\{f-q: q \in S\}$ is a finite dimensional affine subspace of $C^{(l)}(I)$. Since the finite dimensional p-norm is strictly convex on R^K then there exists a unique element in K with minimum p-norm. Now suppose q_1 and q_2 both minimize $N(f-q)$. Then $\Phi(f-q_1) = \Phi(f-q_2)$. But then by linearity of Φ,

$$\Phi(q_1 - q_2) = 0$$

so that

$$(q_1 - q_2)^{(l)}(x_j) = 0, \qquad j = 1, ..., k.$$

But q_1, $q_2 \in S$ implies that $(q_1 - q_2)^{(i)}(x_j) = 0$, $i = 0, ..., l-1$, $j = 1, ..., k$. But m is an ETS of dimension $n+1 < lk+1$ so that $q_1 = q_2$. ∎

Putting together our previous results we have the main result for the case $1 < p < \infty$.

THEOREM 5. *Let $f \in C^{l+1}[I]$ and $1 < p < \infty$ be fixed. Then the net $q_h \to q_0$ uniformly on I as $h \to 0^+$ where q_0 is the unique member of S solving* (6.1).

Proof. Let $\{q_{h_v}\}$ be an arbitrary subsequence of $\{q_h\}$. Then there is a further subsequence (which we do not relabel) such that $q_{h_v} \to q_0$. Then $q_0 \in S$ (Theorem 2) and q_0 solves (6.1) (Theorem 3). By Theorem 4, q_0 is unique. Thus $\{q_h\}$ has a unique cluster point q_0 and hence $q_h \to q_0$ as $h \to 0^+$. ∎

Remark 1. The main result [4, Theorem 2.9, p. 43] follows from our results as follows.

Let $k=2$, $x_1=-1$, x_2-1, and $f \in C^n[-1,1]$. Then the best L^2 local approximation q_0 to f from $M=\Pi_{2n}$ is uniquely determined by the interpolation conditions:

(i) $q_0^{(i)}(\pm 1) = f^{(i)}(\pm 1)$, $i=0, 1, ..., n-1$.

(ii) $q_0^{(n)}(1) + (-1)^n q_0^{(n)}(-1) = f^{(n)}(1) + (-1)^n f^{(n)}(-1)$.

Proof. Since $q_0 \in S$, (i) follows immediately. To obtain (ii), first note that q_0 is the unique minimizer of

$$\Phi(q) = [f^{(n)}(-1) - q^{(n)}(-1)]^2 + [f^{(n)}(1) - q^{(n)}(1)]^2$$

$$\text{as} \quad q \text{ varies over } S.$$

But $q \in S$ implies $q = q_{2n-1}(x) + c(x+1)^n (x-1)^n$ for some constant where $q_{2n-1}(x)$ is the Hermite interpolating polynomial of degree $2n-1$ for f using the points -1 and 1. This gives $q^{(n)}(x) = q_{2n-1}^{(n)} + n!\, c[(x+1)^n + (x-1)^n]$ so that $(\partial q^{(n)}/\partial c)(\pm 1) = (\pm 1)^n n!\, 2^n$. Thus viewing Φ as a function of c, q_0 is characterized by the condition $(\partial \Phi/\partial c)(c_0) = 0$ where c_0 is the constant associated with q_0. Applying this condition and simplifying yields (ii). ∎

Remark 2. A similar characterization may be obtained for the 3-point local L^2 approximation q_0 to f from $\prod_{3n}$. Indeed, q_0 is characterized by the conditions:

(i) $q_0^{(i)}(x) = f^{(i)}(x)$, $x = -1, 0, 1$ and $i = 0, 1, ..., n-1$.

(ii) $[f^{(n)}(-1) - q_0^{(n)}(-1)] + [f^{(n)}(1) - q_0^{(n)}(-1)] = ((-1)^{n+1}/2^n)$ $[f^{(n)}(0) - q_0^{(n)}(0)]$.

7

The Case $P = 1$

Simple examples show (see [7, p. 42]) that the best L^1 approximation to a continuous function from $\prod_n$ on a closed set containing two disjoint intervals is not necessarily unique and that given a net $\{q_h\}$ of best L^1

approximations to f, then the cluster points of this net may be infinite in number. Thus the methods applied in the case $1 < P < \infty$ yield the following weaker version of Theorem 5.

THEOREM 6. *Given $f \in C^{l+1}[I]$, let $q_h(f)$ be a best L^1 approximation to f from M on I_h. Then $\{q_h(f)\}$ is uniformly bounded and every cluster point q_0 of $\{q_h\}$ as $h \to 0^+$ is in S and minimizes $N(f-q)$ as q varies over S.*

The Case $P = \infty$

Most of the analysis that goes into this case is analogous to that for $1 < P < \infty$ and we refer the reader to [7] for the complete details. We shall only examine the uniqueness question for the minimization of $N(f-q)$ as q ranges over S in detail since the infinity norm is not strictly convex. We begin with a standard definition.

DEFINITION. Let X be a compact Hausdorff space. We say L is a Haar subspace of $C(X)$ of dimension r on X if and only if L is a subspace and zero is the only function in L that has r or more roots in X.

Let $X = \{x_1, ..., x_k\}$ and $H = \{q^{(l)}\colon\ q^{(i)}(x_j) = 0,\ i = 0,\ 1, ..., l-1;\ j = 1, ..., k;\ q \in M\}$ where M is an ETS of dimension $n+1 = lk + r$ on $I \supset X$.

LEMMA 6. *H is a Haar subspace of dimension r on X.*

Proof. Clearly, H is a subspace of $C(X)$ = the space of all functions on X. Suppose H is not Haar. Then we can find a nonzero element $h \in H$ such that h has at least r roots in X. Let $q \in M$ be such that $q^{(l)} = h$. Then

$$q^{(i)}(x_j),\ i = 0,\ 1, ..., l-1;\ j = 1, ..., k.$$

Moreover, $q^{(l)}$ has at least r zeros in X which means that q has at least $kl + r$ zeros in X including multiplicities. But $q \in M$ and M is of dimension $n+1 = lk + r$ and is an ETS so $q \equiv 0$ which is a contradiction. ∎

Recall now that for $P = \infty$, $N(f-q) = \max_{x \in X} |f^{(l)}(x) - q^{(l)}(x)|$.

THEOREM 7. *If $f \in C^{(l+1)}(I)$, there is a unique $q_0 \in S$ such that*

$$N(f - q_0) = \min_{q \in S} N(f-q).$$

Proof. Suppose q_1 and q_2 are both minimizers of $N(f-q)$ as q ranges over S. Then $N(f-q_1) = N(f-q_2)$ and

$$(q_1^{(i)} - q_2^{(i)})(x_j) = 0,\ i = 0,\ 1, ..., l-1,\ j-1, ..., k, \text{ so that } h \equiv q_1^{(l)} - q_2^{(l)} \text{ is a member of } H. \tag{7.1}$$

Now since $\min_{q \in S} N(f-q) = \min_{h \in H} \max_{x \in X} |f^{(l)}(x) - q_2^{(l)}(x) - h(x)|$, then this minimum occurs only when $h \equiv 0$ by the Haar property of H. Thus $q_1^{(l)} = q_2^{(l)}$ and so using (7.1) we conclude $q_1 = q_2$. ∎

Remark. Before stating the local best approximation theorem for the uniform case we note that for each $h > 0$ the best uniform approximation q_h on I_h is uniquely defined if M is a Haar space on I such that I_h is a compact subset of I containing at least $N+1$ points.

THEOREM 8.1. *Let $f \in C^{l+1}[I]$ and $P = \infty$. Then the net $q_h \to q_0$ uniformly on I as $h \to 0^+$ where q_0 is the unique minimizer of $N(f-q)$ as q ranges over S.*

Concluding Remarks. During the analysis presented in this paper, we have always assumed that each interval was of the form $[x_j, x_j + h]$, $j = 1, ..., k$. The only crucial aspect of this is that x_j be in the interval and that the length of the interval is the same for all j. However, if we allow the intervals length to vary (but shrink to zero as $h \to 0$) the limiting q_0 may change. In a two point approximation problem, for example, if the intervals are $[x_1, x_1 + h]$ and $[x_2, x_2 + 2h]$ then the limiting q_0 will minimize a weighted functional N with weights 1/3 and 2/3. Thus, the uniqueness of q_0 also depends on the way in which we shrink the intervals. This phenomenon has been noted by Chui *et al.* also in the more general situation of multivariate local approximation [6].

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